NEW RECURRENT INEQUALITY ON A CLASS OF VERTEX FOLKMAN NUMBERS

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Let G be a graph and $V(G)$ be the vertex set of G. Let a_1 ,..., a_r be positive integers, $m = \sum_{i=1}^{r}(a_i-1)+1$ and $p = \max\{a_1, \dots, a_r\}$. The symbol $G \to \{a_1, \dots, a_r\}$ denotes that in every r-coloring of $V(G)$ there exists a monochromatic a_i-clique of color i for some $i = 1, \dots, r$. The vertex Folkman numbers $F(a_1, \dots, a_r; m-1) = \min\{|V(G)| : G \to (a_1 \dots a_r)$ and $K_{m-1} \not\subseteq G\}$ are considered. In this paper we improve the known upper bounds on the numbers $F(2, 2, p; p+1)$ and $F(3, p; p+1)$.

1 Introduction

We consider only finite, non-oriented graphs without loops and multiple edges. We call a p-clique of the graph G a set of p vertices, each two of which are adjacent. The largest positive integer p, such that the graph G contains a p-clique is denoted by $cl(G)$. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of the graph G respectively. The symbol K_n denotes the complete graph on n vertices.
Let G_1 and G_2 be two graphs without common vertices. We denote by $G_1 + G_2$ the graph G for which $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup E'$, where $E' = \{[x, y] \mid x \in V(G_1), y \in V(G_2)\}$.

Definition. Let $a_1, \dots, a_r$ be positive integers. We say that the r-coloring

$$V(G) = V_1 \cup \ldots \cup V_r \ , \ V_i \cap V_j = \emptyset \ , \ i \neq j,$$

of the vertices of the graph G is $(a_1, \dots, a_r)$-free, if V_i does not contain an a_i-clique for each $i \in \{1, \dots, r\}$. The symbol $G \to (a_1, \dots, a_r)$ means that there is not an $(a_1, \dots, a_r)$-free coloring of the vertices of G.

We consider for arbitrary natural numbers $a_1, \dots, a_r$ and q

$$H(a_1, \dots, a_r; q) = \{G : G \to (a_1, \dots, a_r) \text{ and } cl(G) < q\}.$$

The vertex Folkman numbers are defined by the equality

$$F(a_1, \dots, a_r; q) = \min\{|V(G)| : G \in H(a_1, \dots, a_r; q)\}.$$

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It is clear that $G \to (a_1, \dots, a_r)$ implies $cl(G) \geq \max\{a_1, \dots, a_r\}$. Folkman [1] proved that there exists a graph G such that $G \to (a_1, \dots, a_r)$ and $cl(G) = \max\{a_1, \dots, a_r\}$. Therefore

(1) $$F(a_1, \dots, a_r; q) \text{ exists if and only if } q > \max\{a_1, \dots, a_r\}.$$

If $a_1, \dots, a_r$ are positive integers, $r \geq 2$ and $a_i = 1$ then it is easy to see that

(2) $$G \to (a_1, \dots, a_r) \Leftrightarrow G \to (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_r).$$

It is also easy to see that for an arbitrary permutation $\varphi \in S_r$ we have

$$G \to (a_1, \dots, a_r) \Leftrightarrow G \to (a_{\varphi(1)}, \dots, a_{\varphi(r)}).$$

That is why

(3) $$F(a_1, \dots, a_r; q) = F(a_{\varphi(1)}, \dots, a_{\varphi(r)}), \text{ for each } \varphi \in S_r$$

According to (2) and (3) it is enough to consider just such numbers $F(a_1, \dots, a_r; q)$ for which

(4) $$2 \leq a_1 \leq \dots \leq a_r.$$

For arbitrary positive integers $a_1, \dots, a_r$ define:

(5) $$p = p(a_1, \dots, a_r) = \max\{a_1, \dots, a_r\};$$

(6) $$m = 1 + \sum_{i=1}^{r}(a_i - 1)$$

It is easy to see that $K_m \to (a_1, \dots, a_r)$ and $K_{m-1} \nrightarrow (a_1, \dots, a_r)$. Therefore

$$F(a_1, \dots, a_r; q) = m, \text{ if } q > m.$$

In [4] it was proved that $F(a_1, \dots, a_r; m) = m + p$, where m and p are defined by the equalities (5) and (6). About the numbers $F(a_1, \dots, a_r; m-1)$ we know that $F(a_1, \dots, a_r; m-1) \geq m + p + 2$, $p \geq 2$ and

(7) $$F(a_1, \dots, a_r; m-1) \leq m + 3p \text{ , [3].}$$

The exact values of all numbers $F(a_1, \dots, a_r; m-1)$ for which $\max\{a_1, \dots, a_r\} \leq 4$ are known. A detailed exposition of these results was given in [8]. We must add the equality $F(2, 2, 3; 4) = 14$ obtained in [2] to this exposition. We do not know any exact values of $F(a_1, \dots, a_r; m-1)$ in the case when $\max\{a_1, \dots, a_r) \geq 5$.
According to (1), $F(a_1, \dots, a_r; m-1)$ exists exactly when $m \geq p + 2$. In this paper we shall improve inequality (7) in the boundary case when $m = p + 2$, $p \geq 5$. From the equality $m = p + 2$ and (4) it easily follows that there are two such numbers only: $F(2, 2, p; p+1)$ and $F(3, p; p+1)$. It is clear that from $G \to (3, p)$ it follows $G \to (2, 2, p)$. Therefore

(8) $$F(2, 2, p; p+1) \leq F(3, p; p+1).$$

The inequality (7) gives us that:

$$F(3,p;p+1) \le 4p+2; \tag{9}$$

$$F(2,2,p:p+1) \le 4p+2. \tag{10}$$

Our goal is to improve the inequalities (9) and (10). We shall need the following

Lemma. *Let G_1 and G_2 be two graphs such that*

$$G_1 \to (a_1,\dots,a_{i-1},a_i',a_{i+1},\dots,a_r) \tag{11}$$

and

$$G_2 \to (a_1,\dots,a_{i-1},a_i'',a_{i+1},\dots,a_r). \tag{12}$$

Then

$$G_1+G_2 \to (a_1,\dots,a_{i-1},a_i'+a_i'',a_{i+1},\dots,a_r). \tag{13}$$

Proof. Assume that (13) is wrong and let

$$V_1 \cup \ldots \cup V_r\ ,\ V_i \cap V_j = \emptyset\ ,\ i \neq j$$

be a $(a_1,\dots,a_{i-1},a_i'+a_i'',a_{i+1},\dots,a_r)-$free $r-$coloring of $V(G_1+G_2)$.Let $V_i' = V_i \cap V(G_1)$ and $V_i'' = V_i \cap V(G_2)$, for $i=1,\dots,r$. Then $V_1' \cup \ldots \cup V_r'$ is an $r-$coloring of $V(G_1)$, such that V_j does not contain an a_j-clique, $j \neq i$. Thus from (11) it follows that V_i' contains an $a_i'-$clique. Analoguously from the $r-$colouring $V_1'' \cup \ldots \cup V_r''$ of $V(G_2)$ it follows that V_i'' contains an a_i''-clique.Therefore $V_i = V_i' \cup V_i''$ contains a $(a_i'+a_i'')-$clique,which contradicts the assumption that $V_1 \cup \ldots \cup V_r$ is a $(a_1,\dots,a_{i-1},a_i'+a_i'',a_{i+1},\dots,a_r)-$ free $r-$coloring of $V(G_1+G_2)$. This contradiction proves the Lemma. $\square$

2 Results

The main result in this paper is the following

Theorem. *Let $a_1 \le \ldots \le a_r$, $r \ge 2$ be positive integers and $a_r = b_1+\ldots+b_s$, where b_i are positive integers ,too and $b_i \ge a_{r-1}$, $i=1,\dots,s$. Then*

$$F(a_1,\dots,a_r;a_r+1) \le \sum_{i=1}^{s} F(a_1,\dots,a_{r-1},b_i;b_i+1). \tag{14}$$

Proof. We shall prove the Theorem by induction on s. As the inductive step is trivial we shall just prove the inductive base $s=2$. Let G_1 and G_2 be two graphs such that $cl(G_1)=b_1$ and $cl(G_2)=b_2$, $a_r=b_1+b_2$, $b_1 \ge a_{r-1}$, $b_2 \ge a_{r-1}$ and

$$G_1 \to (a_1,\dots,a_{r-1},b_1)\ ,\ |V(G_1)| = F(a_1,\dots,a_{r-1},b_1;b_1+1)$$
$$G_2 \to (a_1,\dots,a_{r-1},b_2)\ ,\ |V(G_2)| = F(a_1,\dots,a_{r-1},b_2;b_2+1).$$

According to the Lemma, $G_1 + G_2 \to (a_1, \dots, a_{r-1}, a_r)$. As $cl(G_1 + G_2) = cl(G_1) + cl(G_2) = b_1 + b_2 = a_r$, we have

$$F(a_1, \dots, a_r; a_r + 1) \le |V(G_1 + G_2)| = |V(G_1)| + |V(G_2)|.$$

From this inequality (14) trivially follows when $s = 2$ and hence , for arbitrary s, as explained above. The Theorem is proved. $\square$

We shall derive some corollaries from the Theorem. Let $p \ge 4$ and $p = 4k + l,\ \ 0 \le l \le 3$. Then from (14) it easily follows that

$$F(3, p; p + 1) \le (k - 1)F(3, 4, ; 5) + F(3, 4 + l; 5 + l) \tag{15}$$

$$F(2, 2, p; p + 1) \le (k - 1)F(2, 2, 4; 5) + F(2, 2, 4 + l; 5 + l). \tag{16}$$

From (15), (9) ($p = 5, 6, 7$) and the equality $F(3, 4; 5) = 13$, [6] we obtain

Corollary 1. *Let $p \ge 4$. Then:*

$$F(3, p; p + 1) \le \frac{13p}{4} \quad \textit{for} \quad p \equiv 0 \mod 4;$$
$$F(3, p; p + 1) \le \frac{13p + 23}{4} \quad \textit{for} \quad p \equiv 1 \mod 4;$$
$$F(3, p; p + 1) \le \frac{13p + 26}{4} \quad \textit{for} \quad p \equiv 2 \mod 4;$$
$$F(3, p; p + 1) \le \frac{13p + 29}{4} \quad \textit{for} \quad p \equiv 3 \mod 4.$$

From (16), the equality $F(2, 2, 4; 5) = 13$, [7], the inequality (10) ($p = 5$) and the inequalities $F(2, 2, 6; 7) \le 22$, [9]; $F(2, 2, 7; 8) \le 28$, [9] we obtain

Corollary 2. *Let $p \ge 4$. Then*

$$F(2, 2, p; p + 1) \le \frac{13p}{4} \quad \textit{for} \quad p \equiv 0 \mod 4;$$
$$F(2, 2, p; p + 1) \le \frac{13p + 23}{4} \quad \textit{for} \quad p \equiv 1 \mod 4;$$
$$F(2, 2, p; p + 1) \le \frac{13p + 10}{4} \quad \textit{for} \quad p \equiv 2 \mod 4;$$
$$F(2, 2, p; p + 1) \le \frac{13p + 21}{4} \quad \textit{for} \quad p \equiv 3 \mod 4.$$

We conjecture that the following inequalities hold:

$$F(3, p; p + 1) \le \frac{13p}{4} \quad \text{for } p \ge 4; \tag{17}$$

$$F(2, 2, p; p + 1) \le \frac{13p}{4} \quad \text{for } p \ge 4. \tag{18}$$

From the Theorem it follows that

$$F(3, p; p + 1) \le F(3, p - 4; p - 3) + F(3, 4; 5), p \ge 8; \tag{19}$$

(20) $$F(2,2,p;p+1) \leq F(2,2,p-4;p-3) + F(2,2,4;5), p \geq 8.$$

From $F(3,4;5) = 13$ (see [6]) and (19) we obtain

Corollary 3. *If the inequality (17) holds for $p = 5$, 6 and 7, then (17) is true for every $p \geq 4$.*

From $F(2,2,4;5) = 13$, (see [7]) and from (20) it follows

Corollary 4. *If the inequality (18) holds for $p = 5$, 6 and 7 then (18) is true for every $p \geq 4$.*

At the end in regard with (8) we shall pose the following

Problem. *Is there a positive integer p, for which $F(2,2,p;p+1) \neq F(3,p;p+1)$?*

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